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The Knapsack Problem Approach in Solving Partial Hedging Problems of Options

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Abstract

This thesis introduces a new approach for studying the problem of optimal partial hedging of both European and American options in a finite and complete discrete-time market model. We show how certain partial hedging problems, that have been treated earlier using other methods, can alternatively be reduced to different types of knapsack problems, which are a well-known subject in the field of linear programming. We also pose two new partial hedging problems for American options and solve them as knapsack problems.

The main focus is on hedging problems, where optimality is measured in terms of success probability. In these cases the problems are reduced to knapsack problems of a 0-1 type, which in turn can be solved approximately using a so called greedy algorithm. We show how the greedy algorithm can be implemented efficiently in a binomial model.

Sammanfattning

Avhandlingen presenterar ett nytt sätt att betrakta det optimala partiella skyddsproblemet av europeiska och amerikanska optioner på en ändlig och komplett marknad i diskret tid. Det visas hur vissa skyddsproblem, som tidigare behandlats med hjälp av andra metoder, kan alternativt reduceras till olika typer av kappsäcksproblem (knapsack problems), som är allmänt kända inom linjär programmering. Vi formulerar också två nya partiella skyddsproblem för amerikanska optioner och löser dem som kappsäcksproblem.

Huvudfokus är på skyddsproblem, där optimaliteten mäts i form av framgångssannolikheten. I dessa fall reduceras problemen till 0-1 kappsäcksproblem, som i sin tur kan lösas approximativt med hjälp av en så kallad girig algoritm (greedy algorithm). Vi visar hur man kan implementera algoritmen effektivt i en binomialmodell.

Foreword and acknowledgements

"In a finite and complete discrete-time market model the quantile hedging problem can be written as a 0-1 knapsack problem." This rather harmless discovery in the fall of 2008 caused a snowball effect that first resulted in a licentiate thesis, then gave rise to two accepted papers, and has now finally ended in this newly finished doctoral thesis.

At this point I naturally want to thank my advisor, Paavo Salminen. It has been a pleasure to co-operate with him. I appreciate his patience, which he has shown by letting me carry out my research without stressful deadlines. His suggestions have also helped me to improve the papers and this thesis remarkably.

I also want to thank my advisor as well as Göran Högnäs for allowing me to work part-time. I understand that employing someone part-time increases the risk of slower progress and somewhat longer graduation times. I am glad that their confidence in me did not turn out to be totally in vain as this completed thesis indicates.

Finally, I think that the whole staff at the Department of Mathematics deserves special thanks for a very pleasant working atmosphere. The Finnish Graduate School in Stochastics and Statistics, the Academy of Finland and Waldemar von Frenckell's foundation are acknowledged for their financial support.

Åbo, August 2012
Peter Lindberg

List of included articles

This thesis consists of an introductory part and the following papers:

- Lindberg, P.: Optimal partial hedging in a discrete-time market as a knapsack problem. Mathematical Methods of Operations Research 72: 433–451 (2010)
- Lindberg, P.: Greedy algorithm for European lookback options, Supplementary material to the article "Optimal partial hedging in a discrete-time market as a knapsack problem" (2012)
- Lindberg, P.: Optimal partial hedging of an American option: shifting the focus to the expiration date. Mathematical Methods of Operations Research 75: 221–243 (2012)

The introductory part presents and explains the results in the papers above.

1 Introduction: Partial hedging of options

In mathematical finance, an option is a contract that gives its holder the right, but not the obligation, to buy or sell a certain asset at a given price and time in future from/to the option writer. If the option is European, the holder can exercise the option (i.e. force the writer to a transaction) only at a given, in contract specified time point, which is called the expiration date. In the American case the option can be exercised at any time before or at the expiration date. The seller or writer of the option gets an initial payment in form of the option price and binds herself to meet the potential obligation that the option may cause in future. It is clear that the writer is exposed to a risk, which raises a question of how she can protect herself against this risk.

It is well-known that in arbitrage-free and complete market models any option has a unique arbitrage-free price at which it should be sold. This price is the least amount at which the writer of the option can create a perfect hedging strategy, an investment strategy that in any future scenario generates enough wealth to cover the payment that the potential exercise of the option may force her to make. Thus, following the perfect hedging strategy neutralizes the risk completely.

However, in some cases the writer may not be willing to use the whole option price for hedging purposes. Instead, she may want to accept some amount of risk in order to reduce the initial cost of the hedging strategy. What could be a reasonable way of creating such a partial hedging strategy? This question gives rise to interesting optimization problems.

Our key references as regards partial hedging problems are the papers of Föllmer and Leukert [7], [8], which have been an inspiration for and quoted in several subsequent articles (see also Föllmer and Schied [9]). In the papers above the authors look for an optimal partial hedging strategy for a European option in a continuous time model under a cost constraint. In [7] the optimality is measured in terms of success probability (i.e. the probability that the hedging strategy will cover the option at the expiration date), which is to be maximized. This approach is referred to as quantile hedging. In [8] the quantity to be minimized is the expectation of the potential shortfall, that is weighted by a convex loss function. The solution methods are based on either applications of Neyman-Pearson lemma or convex duality.

In this thesis we concentrate mainly on hedging problems, where the goal is to optimize the trade-off between price and success probability. Moreover, we study the problem in a simpler model, a discrete-time model that is both finite and complete. We show that in this setting the partial hedging problem can be reduced to a knapsack problem and that an approximately optimal solution can be obtained using a so called greedy algorithm. We will account for the market model as well as the concepts of knapsack problem and greedy algorithm in Sections 2 and 3. The connection between hedging problems and knapsack problems in the European case is established in Section 4.

Perhaps the most important contribution of our research is that we are able to apply our methods also for American options. Optimal partial hedging has been widely studied for European options, but there are only a few papers addressing the problem in the American setting (see Dolinsky and Kifer [3], [4], Mulinacci [17], Novikov [18], Pérez-Hernández [19] and Treviño [23]). In Section 5 we will present two previously unconsidered partial hedging problems for American options that can be reduced to and solved as knapsack problems.

In the concluding Section 6 we study the greedy algorithm in a binomial model. We show how the algorithm can be implemented efficiently in case of simple options and lookback options of European type.

2 Market model

In papers [12] and [14] we study partial hedging problems of options in a discrete-time market model that is finite, arbitrage-free and complete. Such a market model is used e.g. in Lamberton and Lapeyre [11], pp. 15–49.

The model is built on a finite filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, \mathbb{P})$, where $\Omega = \{\omega_1, \ldots, \omega_n\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$ and $\mathbb{P}(\{\omega_i\}) > 0$ for all $i = 1, \ldots, n$.[1] Here $\mathcal{P}(\Omega)$ denotes the set of all subsets of Ω.

The model is discrete-time: the asset prices are quoted and transactions are made only at times $0, 1, \ldots, T$, not continuously. We set T to be equal to the expiration date of a European or an American option which we want to hedge. The set Ω is the set of all alternatives or scenarios according to which the market can evolve from time 0 to time T. The model is finite, meaning that the number of elements in Ω is finite. The filtration $\{\mathcal{F}_t\}_{t=0}^T$ is used to model the gradually increasing information of the market evolvement.

The prices of the $d+1$ assets on the market follow a $d+1$-dimensional non-negative $\{\mathcal{F}_t\}$-adapted stochastic process $\{S_t\}_{t=0}^T$, where

$$S_t = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})$$

[1]In this thesis we omit the definitions of elementary stochastic concepts. A good supplementary reading is the book of David Williams [24].

and $S_t^{(j)}$ is the price of asset j at time t. However, here we will follow the approach in Föllmer and Schied [9] and present all values in units of the *numéraire* asset $S^{(0)}$, whose value therefore is assumed to be strictly positive at all times. The discounted price of asset j at time t (i.e. its value in units of the numéraire asset $S^{(0)}$) is given by $X_t^{(j)} := S_t^{(j)}/S_t^{(0)}$ and the corresponding price process is $\{X_t\}_{t=0}^T$, where

$$X_t = (1, X_t^{(1)}, \ldots, X_t^{(d)}).$$

A trading strategy is an $\mathbb{R}^{d+1}$-valued $\{\mathcal{F}_t\}$-predictable stochastic process $\xi = \{\xi_t\}_{t=0}^T$, where

$$\xi_t = (\xi_t^{(0)}, \xi_t^{(1)}, \ldots, \xi_t^{(d)})$$

and $\xi_t^{(j)}$ is the quantity of asset j in the portfolio at time t. Here predictability means, as in [11], that ξ_0 is $\mathcal{F}_0$-measurable and ξ_t is $\mathcal{F}_{t-1}$-measurable for all $t \geq 1$. The value process $V = \{V_t\}_{t=0}^T$ of ξ is defined through

$$V_t = \xi_t \cdot X_t = \sum_{j=0}^{d} \xi_t^{(j)} X_t^{(j)}.$$

A strategy ξ is called *self-financing* if

$$\xi_t \cdot X_t = \xi_{t+1} \cdot X_t$$

for all $t = 0, \ldots, T-1$. The self-financing condition states that the portfolio is adjusted at each time point without inserting new or withdrawing already existing capital. A self-financing strategy ξ is called *admissible* if its value process satisfies $V_t \geq 0$ for all $t = 0, \ldots, T$.

We require that our market model is arbitrage-free, which means that for any admissible strategy with $V_0 = 0$ it has to hold that also $V_T = 0$. This means that with zero initial payment it is impossible to obtain an opportunity for a positive profit without any fear of loss.

We represent a European option through an $\mathcal{F}_T$-measurable function $H \geq 0$, which denotes the discounted payoff (or value) of the option at maturity. The market model is assumed to be complete, which means that any European option can be replicated, i.e. for any discounted payoff H there is a self-financing strategy ξ^H with value process $\{H_t\}_{t=0}^T$ so that $H_T = H$. The strategy ξ^H is called the replicating strategy or the perfect hedging strategy of the European option with discounted payoff H.

Since we assume that the market is both arbitrage-free and complete, we know that there exists a unique equivalent martingale measure, denoted by $\mathbb{P}^*$. It holds that the value process of a self-financing strategy is a $\mathbb{P}^*$-martingale. This property yields together with the no-arbitrage property that the initial price of ξ^H equals the price of the option and is given by $H_0 = \mathbb{E}^*(H)$.

As regards American options, we use Z_t to denote the discounted payoff of an American option at time t, i.e. the amount in units of the numéraire asset that the holder of the option gets if she chooses to exercise the option at time t. The process $\{Z_t\}_{t=0}^T$ is assumed to be non-negative and $\{\mathcal{F}_t\}$-adapted. The value process of the option is given by the process $\{U_t\}_{t=0}^T$ defined through

$$\begin{cases} U_T &= Z_T \\ U_t &= \max\{Z_t, \mathbb{E}^*(U_{t+1}|\mathcal{F}_t)\}, \quad t = 0, 1, \dots, T-1. \end{cases}$$

The process $\{U_t\}_{t=0}^T$ is a $\mathbb{P}^*$-supermartingale and it is called the Snell envelope of the process $\{Z_t\}_{t=0}^T$. It has the unique Doob decomposition $U_t = M_t - D_t$, where $\{M_t\}_{t=0}^T$ is a $\mathbb{P}^*$-martingale and $\{D_t\}_{t=0}^T$ is a non-decreasing, $\{\mathcal{F}_t\}$-predictable process with $D_0 = 0$. There exists an admissible strategy ξ^M whose value process is equal to $\{M_t\}_{t=0}^T$. This strategy can be used as a perfect hedging strategy for the American option described by the sequence $\{Z_t\}_{t=0}^T$. The initial price of ξ^M is $U_0 = M_0$ in units of the numéraire asset.

We use $\mathcal{T}_{\{0,T\}}$ to denote the set of all stopping times taking values in $\{0, 1, \dots, T\}$. If τ^* is an optimal stopping time, i.e. if

$$\mathbb{E}^*(Z_{\tau^*}) = \sup_{\tau \in \mathcal{T}_{\{0,T\}}} \mathbb{E}^*(Z_\tau),$$

then $Z_{\tau^*} = U_{\tau^*} = M_{\tau^*}$, i.e. $D_{\tau^*} = 0$. For instance, $\tau_0 := \inf\{t \geq 0 | U_t = Z_t\}$ is an optimal stopping time. In fact, τ_0 is the earliest optimal stopping time.

3 The knapsack problem

Knapsack problems are commonly known in the field of linear programming. The knapsack problem can be illustrated as follows (see e.g [2], p. 273 or [15], p. 1): A traveller has to fill a knapsack of a certain size C by selecting some of n items having sizes w_i, $i = 1, \dots, n$, respectively. The "gain" given by the items is measured with numbers g_i, $i = 1, \dots, n$, respectively. The traveller wants to select objects that give her the maximal total "gain" under the constraint that the total size of the chosen objects will not exceed the knapsack size C. A possible decision is modeled by an n-dimensional vector

x whose elements satisfy

$$x_i = \begin{cases} 1 & \text{if object } i \text{ is selected} \\ 0 & \text{otherwise.} \end{cases}$$

Mathematically the problem is formulated as follows:

Problem 1. *Find an n-dimensional vector $x^* = (x_1^*, \ldots, x_n^*)$ that maximizes*

$$\sum_{i=1}^{n} g_i x_i$$

under constraints

$$\sum_{i=1}^{n} w_i x_i \leq C$$

and

$$x_i = 0 \text{ or } 1, \;\; i = 1, \ldots, n. \tag{1}$$

The vector x^* is called the *optimal solution vector.* We use

$$z^* := \sum_{i=1}^{n} g_i x_i^* \tag{2}$$

to denote the *optimal solution value.* Moreover, the set of items corresponding to the optimal solution vector is called the *optimal solution set.*

Problem 1 is referred to as a *0-1 knapsack problem* because of the condition (1). By modifying this condition we get different types of knapsack problems. In particular, in this thesis we will frequently refer to the *continuous knapsack problem*, where the condition (1) is replaced by

$$0 \leq x_i \leq 1, \;\; i = 1, \ldots, n.$$

This corresponds to a case where the traveller of our illustration can choose any fractions of objects to her knapsack. We will also consider the case where (1) is replaced by

$$x_i \leq b_i, \;\; i = 1, \ldots, n$$

with given numbers $b_i \geq 0$, $i = 1, \ldots, n$. In this case we call the problem an *unbounded knapsack problem*, since x_i may attain arbitrarily large negative values. For other types of knapsack problems, we refer to Martello and Toth [15], p. 1–5.

As regards solving the above-mentioned problems, the two last-mentioned problems have an easy solution. Assume that the items are ordered so that

$$\frac{g_1}{w_1} \geq \frac{g_2}{w_2} \geq \ldots \geq \frac{g_n}{w_n} \tag{3}$$

and define a critical element

$$s := \min\left\{ j : \sum_{i=1}^{j} w_i > C \right\}. \tag{4}$$

The solution to the continuous knapsack problem is given by vector x', where

$$x'_i = \begin{cases} 1, & i = 1, \ldots, s-1 \\ \dfrac{C - \sum_{j=1}^{s-1} w_j}{w_s}, & i = s \\ 0, & i = s+1, \ldots, n \end{cases} \tag{5}$$

(consult e.g. Dantzig [2], Kellerer et al. [10] or Martello and Toth [15] for a proof). The idea is that the traveller chooses consecutively the items that have the best gain over size-ratio as long as she reaches the critical item that no longer fits in the knapsack. Finally, the remaining space is filled with a suitable fraction of this critical item. It can be shown in the same way that the solution to the unbounded knapsack problem is given by

$$\begin{aligned} x_i^\circ &= b_i, \ i = 1, \ldots, n-1 \\ x_n^\circ &= \frac{C - \sum_{j=1}^{n-1} b_j w_j}{w_n}. \end{aligned} \tag{6}$$

The 0-1 knapsack problem in turn is an NP-hard (non-deterministic polynomial time hard) problem. There is a wide range of numerical algorithms that have been developed to solve the problem (see e.g. Kellerer et al. [10], Martello and Toth [15] or Martello et al. [16] for an overview).

An approximately optimal solution x^G for the 0-1 knapsack problem can be obtained by setting $x'_s = 0$ in (5), i.e.

$$x_i^G = \begin{cases} 1, & i = 1, \ldots, s-1 \\ 0, & i = s, \ldots, n. \end{cases}$$

This method is referred to as the greedy algorithm in [10] and [15]. For the solution values z^* in (2),

$$z' := \sum_{i=1}^{n} g_i x'_i$$

and

$$z^G := \sum_{i=1}^{n} g_i x_i^G$$

it clearly holds that

$$0 \leq z^* - z^G \leq z' - z^G \leq x'_s g_s \leq g_{max}, \tag{7}$$

where

$$g_{max} := \max_{1 \leq i \leq n} g_i. \tag{8}$$

Thus, the solution value given by the greedy algorithm falls short of the exact optimal solution value by at most $x'_s g_s$. Note that in case

$$\sum_{j=1}^{s-1} w_j = C \tag{9}$$

we have by (5) that $z^* = z' = z^G$, i.e. the solution given by the greedy algorithm is in fact exact.

We call g_{max} in (8) an *a priori* upper bound for the error, since it does not depend on the critical element s and can therefore be determined before implementing the greedy algorithm. Although g_{max} may in general be too large to give a satisfactory upper bound for the approximation error, we will see in Section 6.1 that it works well, when the greedy algorithm is used in a binomial model.

4 The connection between knapsack problems and partial hedging problems of European options

In this section we show how the knapsack problem theory described in the previous section can be used in solving certain partial hedging problems of European options.

4.1 Quantile hedging

As in Section 2, let H_0 denote the price of a European option with discounted payoff H. Suppose that the investor does not want to use more than v, $v \leq H_0$ to construct a partial hedging strategy for the option. Föllmer and Leukert [7] and Föllmer and Schied [9] consider the following "quantile hedging" problem.

Problem 2. *Find an admissible strategy whose value process V maximizes $\mathbb{P}(V_T \geq H)$ under the constraint $V_0 \leq v$.*

The following result can be found in Föllmer and Schied [9], p. 335.

Theorem 3. *Assume that the set $A^* \in \mathcal{F}_T$ maximizes the probability $\mathbb{P}(A)$ among all sets $A \in \mathcal{F}_T$ satisfying the constraint*

$$\mathbb{E}^*(H \cdot 1_A) \leq v.$$

Then the replicating strategy ξ^* of the option $H^* := H \cdot 1_{A^*}$ solves Problem 2. Moreover,

$$A^* = \{V_T^* \geq H\} \quad \mathbb{P} - a.s.,$$

where V^* is the value process of the strategy ξ^*.

Consequently, the dynamic problem of finding an optimal strategy is reduced to a static problem of finding an optimal "success set" A^*, i.e. a set of scenarios where the option will be covered. Föllmer and Schied [9] do not discuss how the set A^* can be found in general. However, they define a measure

$$d\mathbb{Q} := \frac{H}{\mathbb{E}^*(H)} d\mathbb{P}^*$$

and a level

$$c^* := \inf \left\{ c \geq 0 | \mathbb{Q}\left(\frac{d\mathbb{P}}{d\mathbb{Q}} > c \cdot \mathbb{E}^*(H) \right) \leq \frac{v}{\mathbb{E}^*(H)} \right\},$$

where the density $d\mathbb{P}/d\mathbb{Q}$ stems from the Lebesque decomposition theorem (see [9], p. 405). It is shown, using the Neyman-Pearson lemma, that if the equality

$$\mathbb{Q}(d\mathbb{P}/d\mathbb{Q} > c^* \cdot \mathbb{E}^*(H)) = v/\mathbb{E}^*(H) \tag{10}$$

holds, then $A^* = \{d\mathbb{P}/d\mathbb{Q} > c^* \cdot \mathbb{E}^*(H)\}$ is an optimal set described in Theorem 3.

The knapsack problem approach comes in at this point. As we showed in [12], the problem of finding the set A^* can be described as a 0-1 knapsack problem. Indeed, A^* will be the optimal solution set for Problem 1, where the "items" to choose from are scenarios $\omega_1, \ldots, \omega_n$,

$$g_i := \mathbb{P}(\omega_i) \tag{11}$$

and

$$w_i := \mathbb{Q}(\omega_i) = \frac{\mathbb{P}^*(\omega_i) H(\omega_i)}{\mathbb{E}^*(H)} \tag{12}$$

for all $i = 1, \ldots, n$, and $C = v/\mathbb{E}^*(H)$.

By applying greedy algorithm for this 0-1 knapsack problem we get an approximately optimal solution for Problem 2. Assume that the scenarios $\omega_1, \ldots, \omega_n$ are ordered so that g_i/w_i is decreasing. In case $w_i = 0$, which happens if $H(\omega_i) = 0$, we define $g_i/w_i = +\infty$. Then the replicating strategy of the claim $H1_{\{\omega_i : 1 \leq i \leq s-1\}}$, where s is as in (4), gives an approximately optimal solution for Problem 2. Note the analogy between conditions (10) and (9): if (9) holds, the solution given by the greedy algorithm is, in fact, exact.

By (7) and (11) the success probability attained by the approximately optimal strategy falls short of the maximal possible success probability by at most

$$p_{max} := \max_{1 \leq i \leq n} \mathbb{P}(\omega_i). \tag{13}$$

4.2 The case where the size of the potential shortfall is taken into account

One of the main disadvantages associated with the quantile hedging approach is that it does not take into account the size of the potential shortfall (see e.g. the discussion in Föllmer and Leukert [7], p. 253 and Cvitanić and Karatzas [1], p. 452). That is one of the reasons why we in [14] consider the following modified problem.

Problem 4. *Let $0 \leq \varepsilon \leq 1$ and $c \geq 0$ be given constants. Find a self-financing strategy ξ^{opt} whose value process minimizes V_0 among all self-financing strategies satisfying $\mathbb{P}(V_T \geq H) \geq 1 - \varepsilon$ and*

$$V_t \geq H_t - c \text{ for all } 0 \leq t \leq T, \tag{14}$$

where $\{H_t\}_{t=0}^T$ is the value process of the European option.

Problem 4 takes into account the size of a potential shortfall. Indeed, the condition (14) ensures that

$$(H - V_T)^+ \leq c, \tag{15}$$

i.e. that the discounted shortfall will be bounded above by a constant c. In fact, a self-financing strategy that satisfies (15), satisfies even (14) because of the martingale property.

As we showed in [14], the solution for Problem 4 is the strategy

$$\xi^{opt} := \xi^H - \xi^c + \xi^*, \tag{16}$$

where ξ^H is the perfect hedging strategy of H, ξ^c is determined through

$$\xi_t^c = (c, 0 \dots, 0) \text{ for all } 0 \leq t \leq T \tag{17}$$

and ξ^* is the replicating strategy of the claim $c1_{\complement X}$, where X is the optimal solution set for Problem 1 with scenarios $\omega_1, \dots, \omega_n$ as items,

$$g_i := \mathbb{P}^*(\omega_i),\ w_i := \mathbb{P}(\omega_i) \text{ and } C := \varepsilon. \tag{18}$$

The optimal strategy has two interesting properties: Firstly, we see that if the optimal strategy is used and a shortfall occurs, its size will be *exactly* c. Moreover, the set X does not depend on c. Therefore, the price of the partial hedging strategy ξ^{opt} is a linear function of the allowed shortfall size c. Indeed, by (16) and (17)

$$V_0^{opt} = H_0 - c + c\mathbb{P}^*\left(\complement X\right) = H_0 - \mathbb{P}^*(X)c. \tag{19}$$

Remark 5. *Recall that in the market model described in Section 2 the numéraire asset $S^{(0)}$ may have a stochastic price process. Therefore, the discounted shortfall c does not necessarily correspond to a constant shortfall* in monetary units. *However, if the numéraire asset is assumed to have a deterministic price process (e.g. a bank account or a bond with deterministic interest rate), as is the case for instance in binomial model, then c will correspond to a constant amount $cS_T^{(0)}$ even in monetary units, thus giving the investor a very clear picture of the size of the potential loss.*

The linear relationship holds also for the approximately optimal solution that is given by the greedy algorithm. As we showed in [14], the strategy

$$\xi^{appr.opt} := \xi^H - \xi^c + \xi^G \tag{20}$$

can be used as an approximately optimal strategy. Here ξ^H and ξ^c are as above and ξ^G is the replicating strategy of the claim $c1_{\{\omega_i : s \leq i \leq n\}}$, where the items are ordered according to (3) with parameters (18) and s is as in (4).[2] This time the linear relationship is given by

$$V_0^{appr.opt} = H_0 - \alpha c, \tag{21}$$

where

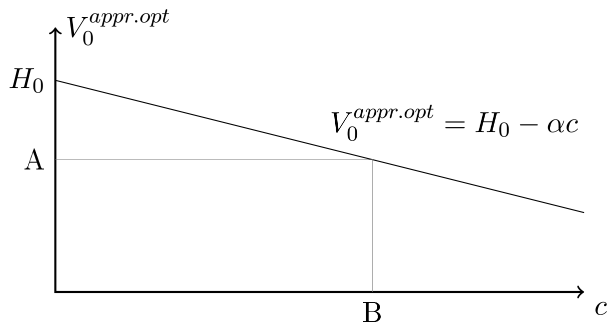
$$\alpha = \sum_{i=1}^{s-1} \mathbb{P}^*(\omega_i).$$

[2]The results in this section are somewhat differently formulated than those in [14]. This is because in [14] the items are in reverse order and the critical element is defined accordingly.

We have that

$$0 \le V_0^{appr.opt} - V_0^{opt} \le c \max_{1 \le i \le n} \mathbb{P}^*(\omega_i). \tag{22}$$

The linear equation (21) is interesting, since it allows the investor to study for a given success probability level the trade-off between the price of the optimal partial hedging strategy and the size of the discounted loss that will be realized in case of a shortfall. Once the investor has decided how big success probability $1-\varepsilon$ she wants the partial hedging strategy to have, and determined the quantity α, she can choose appropriate values for c and $V_0^{appr.opt}$ using equation (21) (see Figure 1).



A = how much the investor is willing to pay for the hedging strategy
B = how much the investor is prepared to lose in case of shortfall

Figure 1: The linear relationship between price and shortfall for a given success probability level.

Remark 6. *Cvitanić and Karatzas [1] and Spivak and Cvitanić [22] solve in a continuous-time setting various hedging problems where the condition* (14) *is required to hold. However, in the papers above this condition is not combined with the specific problem of minimizing the initial cost of a self-financing strategy that covers a European claim with a given probability. As we have seen, in this particular case we obtain a linear relationship between the price of the hedge and the size of the potential shortfall.*

4.3 Other examples

In [12] we discuss also two further hedging problems that can be reduced to knapsack problems. As will be explained below, both problems have been solved earlier by exploiting other methods.

Consider first the following problem, that is posed e.g. in Föllmer and Leukert [8] and Föllmer and Schied [9].

Problem 7. *Find an admissible strategy whose value process V minimizes the expected shortfall $\mathbb{E}[(H - V_T)^+]$ under the constraint $V_0 \leq v$.*

As shown in Föllmer and Schied [9], this problem can be reduced to finding an optimal randomized test. (A randomized test is an $\mathcal{F}_T$-measurable function ψ that satisfies $0 \leq \psi \leq 1$.) Indeed, it is shown that if ψ^* maximizes $\mathbb{E}(H\psi)$ among all randomized tests ψ satisfying $\mathbb{E}^*(H\psi) \leq v$, then the replicating strategy of the modified claim $H\psi^*$ solves Problem 7.

In [9] an optimal randomized test is found by using the generalized Neyman-Pearson lemma. We showed in [12] that in our market model the problem of finding an optimal randomized test can be written and solved as a continuous knapsack problem. Indeed, we have that $\psi^*(\omega_i) = x'_i$, where x' is a solution to the continuous knapsack problem described in Section 3 with $g_i := \mathbb{P}(\omega_i)H(\omega_i)/\mathbb{E}(H)$, w_i as in (12) and $C = v/\mathbb{E}^*(H)$.

As another example consider the following hedging problem.

Problem 8. *Find a self-financing strategy whose value process V minimizes the expected shortfall $\mathbb{E}[(H - V_T)^+]$ under the constraint $V_0 \leq v$.*

The difference to Problem 7 is that this time the value process of the hedging strategy does not need to be non-negative. Favero [5], Favero and Vargiolu [6] and Runggaldier et al. [20] have studied Problem 8 in context of a binomial model and Scagnellato and Vargiolu [21] in a little more general multinomial model. The problem is solved using dynamic programming.

We show in [12] that Problem 8 can alternatively be seen and solved as an unbounded knapsack problem. The solution is the replicating strategy of the claim H°, $H^\circ(\omega_i) = x_i^\circ$, where x° is a solution to the unbounded knapsack problem in Section 3 with $g_i = \mathbb{P}(\omega_i)$, $w_i = \mathbb{P}^*(\omega_i)$, $C = v$ and $b_i = H(\omega_i)$.

Remark 9. *Note that the replicating strategy of the claim H° replicates the original option H in all but one scenario (see equation* (6)*). Such a strategy is in Favero [5] and Favero and Vargiolu [6] called quasi-replicating.*

As we explain in [12], in our finite and complete market model we can with any initial capital v_0 and for any $\omega' \in \Omega$ create a self-financing strategy whose value process satisfies $V_T = H$ on $\Omega \setminus \omega'$ and

$$V_T(\omega') = H(\omega') + \frac{v_0 - \mathbb{E}^*(H)}{\mathbb{P}^*(\omega')}.$$

Thus, if we in Problem 2 drop the non-negativity constraint and search for an optimal strategy in the set of all self-financing strategies, we can conclude that the optimal strategy is a quasi-replicating strategy that replicates the option in all but one scenario that has the least $\mathbb{P}$-probability.

5 Hedging problems for American options

An interesting question that rose during the research that led to this thesis was whether the knapsack problem approach could be applied for hedging problems of *American* options also. It appeared that this indeed is possible. Below we show how two different hedging problems of American options can be reduced to 0-1 knapsack problems. Both problems can be seen as American counterparts of Problem 4.

5.1 A problem that can be solved under a barrier condition

Novikov [18] studies a certain partial hedging problem for an American contingent claim under a so called barrier condition. The barrier condition states that the option holder cannot exercise the option before a given optimal stopping time τ^*, i.e. that the option can be exercised only according to stopping times satisfying $\tau \geq \tau^*$. We denote the set of such stopping times by $\mathcal{T}_{\{\tau^*,T\}}$. Let $0 \leq \varepsilon \leq 1$ and $c \geq 0$ be given constants and Z_t the discounted payoff of the American option at time t. Consider the following hedging problem.

Problem 10. *Find a self-financing strategy that minimizes the initial cost V_0 among all self-financing strategies that satisfy*

$$\mathbb{P}(V_\tau \geq Z_\tau) \geq 1 - \varepsilon \tag{23}$$

and

$$V_\tau \geq Z_\tau - c$$

for all stopping times $\tau \in \mathcal{T}_{\{\tau^*,T\}}$.

The hedging problem that Novikov [18] studies is similar to Problem 10, except that the condition (23) is required to be satisfied for all probability measures belonging to some set of probability measures, *including the martingale measure* $\mathbb{P}^*$. Thus, our approach is not a direct special case.

Even if the barrier condition is a rather inconvenient restriction, it nevertheless makes Problem 10 solvable. Indeed, we are again able to reduce the problem to a 0-1 knapsack problem.

To do this, let $\mathcal{F}_\tau$ be the σ-algebra of all events occurring before or at a given stopping time τ. We call a set $A \in \mathcal{F}_\tau$ an $\mathcal{F}_\tau$-atom if $A \neq \emptyset$ and if each $B \in \mathcal{F}_\tau$ with $B \subseteq A$ satisfies either $B = \emptyset$ or $B = A$. Let $A_1, \ldots, A_k$ be the set of $\mathcal{F}_{\tau^*}$-atoms satisfying $A_i \neq \emptyset$, $A_i \in \mathcal{F}_{\tau^*}$, $\cup_{i=1}^k A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Let now Y be the optimal solution set for Problem 1 with $\mathcal{F}_{\tau^*}$-atoms $A_1, \ldots, A_k$ as items and

$$g_i := \mathbb{P}^*(A_i),\ w_i := \mathbb{P}(A_i) \text{ and } C := \varepsilon.$$

As we show in [14], the optimal solution for Problem 10 is then given by the strategy

$$\xi^M - \xi^c + \xi^*,$$

where ξ^M is the perfect hedging strategy of the American claim, ξ^c is as in (17) and ξ^* is a strategy with value process $\{\mathbb{E}^*(c1_{\complement Y}|\mathcal{F}_t)\}_{t=0}^T$.

Suppose that the greedy algorithm is used to approximate the optimal solution. This time the a priori upper bound for the error in the initial cost is given by

$$\Theta := c \max_{1 \leq i \leq k} \mathbb{P}^*(A_i) \tag{24}$$

(cf. equation (22)). Note that to determine Θ one first has to find out the structure of τ^*, i.e. the knowledge of the market model alone is not enough. As we pointed out in [14], Θ may be very unsatisfactory bound for the error, depending on τ^*.

5.2 A problem where the focus is shifted to the expiration date

Assume that an investor has sold an American option and wants to create a partial hedge for it. However, instead of worrying about the payment that has to be settled at the exercise time the investor decides that at the exercise time she will borrow the amount she has to pay for the option holder by entering a short position of a necessary size in the numéraire asset. Later, at the expiration date of the option, she will pay back this loan, if possible. Consequently, the object of partial hedging is changed from the original option to the loan taken at the exercise time. Suppose that the investor wants to minimize the initial cost at which she can hedge the above-mentioned loan with probability $1 - \varepsilon$ while the potential shortfall is allowed to be at most c numéraire assets.

The whole problem setting described above is new and previously unconsidered. In the literature that we have found regarding partial hedging of

American options (see e.g. Dolinsky and Kifer [3], Mulinacci [17], Novikov [18], Pérez-Hernández [19] and Treviño [23]) the focus is always on the situation at the (stochastic) exercise time of the option. Either the goal is to optimize the investor's position at the exercise time under a cost constraint, or to minimize the initial cost while some conditions are required to be satisfied at the exercise time. As described above, here we shift the focus to the (deterministic) expiration date.

In [14] we explain in detail how the problem above can be modeled and solved. The most complicated part of formulating the problem is to determine in which set the optimal solution should be searched. In the previous sections of this thesis we have been optimizing in the set of self-financing strategies. However, this set is unsuitable for the hedging problem of this section.

For instance, we naturally have to allow that at a given time t and on an $\mathcal{F}_t$-atom A_t the investor may invest differently depending on whether the option has already been exercised or not. The concept of a trading strategy, being an $\{\mathcal{F}_t\}$-predictable stochastic process, does not enable this feature. Therefore, to allow the investor's hedging policy to depend on the option holder's exercise behaviour, we optimize in a certain set of vectors whose elements are self-financing strategies and whose length is the number of stopping times in $\mathcal{T}_{\{0,T\}}$ (cf. Section 2).

We omit the lengthy details here and confine ourselves to explaining the optimal solution. As shown in [14], it is optimal for the investor to do the following: In the beginning, at time zero, she should

- borrow c numéraire assets (this is strategy ξ^c in (16)),
- start following a self-financing strategy that has minimal initial cost while generating the discounted value c at maturity with probability $1-\varepsilon$, thus allowing the investor to return the borrowed numéraire assets with this probability (this is strategy ξ^* in (16) known to us from the European case)
- start following the perfect hedging strategy ξ^M of the American option.

Moreover, the optimal solution suggests that at the exercise time of the option the investor should invest the wealth generated by ξ^M to the numéraire asset.[3] The strategies ξ^c and ξ^* are continued up to the expiration date.

[3] Actually, this wealth could alternatively be used directly to cover the payment for the option holder that needs to be settled at the exercise time. If done so, the investor will not have to take any loan at the exercise time!

Note that the problem is solved without imposing any barrier condition that would restrict the option holder's exercise behaviour, which we had to do in Section 5.1.

The initial payment needed to implement the above-mentioned hedging policy is as in (19), with the exception that H_0 is replaced by U_0, the price of the American option. As in the European case, ξ^* can be approximated by ξ^G, the strategy given by the greedy algorithm, resulting in an analoguous linear relationship that we had in (21), again with $H_0 = U_0$. As in (22), the error in the initial cost is bounded above by

$$\Psi := c \max_{1 \le i \le n} \mathbb{P}^*(\omega_i). \tag{25}$$

Note that Ψ has a clear advantage over the error bound Θ in (24), since it does not depend on the optimal stopping time τ^*. Firstly, this makes Ψ easier to determine. Moreover, Ψ has a given constant value in a market model, whereas Θ takes different (and possibly unsatisfactory large) values for options having different payoffs. As shown in [14], in a binomial model with realistic parameters Ψ is very small (see also Table 6.1 below).

6 Implementing the greedy algorithm in a binomial model

An interesting question is how the theory developed in the previous sections can be applied in practice in a binomial model, which is a widely used finite and complete market model in discrete time.

6.1 The binomial model

In a binomial model with T time steps the market consists of two assets, namely a bond (or a bank account) B and a stock S. For the bond price we assume $B_t = r^t$ for $0 \le t \le T$, where $r \ge 0$ is a constant. Thus, the price process of the bond is totally deterministic. The stock price on the other hand follows the dynamics

$$S_{t+1} = \xi_{t+1} S_t, \quad t = 0, 1, \ldots, T-1$$

where S_0 is the initial value of the stock and $\{\xi_t\}_{t=1}^T$ are i.i.d. random variables taking values in $\{u, d\}$ $(0 < d < r < u)$ with probability law

$$p := \mathbb{P}(\xi_t = u) = 1 - \mathbb{P}(\xi_t = d), \quad t = 1, \ldots, T.$$

The unique equivalent martingale measure $\mathbb{P}^*$ is given by

$$p^* := \mathbb{P}^*(\xi_t = u) = 1 - \mathbb{P}^*(\xi_t = d) = \frac{r-d}{u-d}, \quad t = 1, \ldots, T.$$

In the binomial model $\Omega := \{u, d\}^T$, i.e. Ω is the set of all possible paths that the stock price can take. As regards the probability of a single path, we have

$$\mathbb{P}(\omega) = p^{L(\omega)}(1-p)^{T-L(\omega)}, \tag{26}$$

where the function $L : \Omega \to 0, 1, \ldots, T$ indicates the number of upward moves on a particular path. Similarly, we have

$$\mathbb{P}^*(\omega) = (p^*)^{L(\omega)}(1-p^*)^{T-L(\omega)}. \tag{27}$$

Clearly, both $\mathbb{P}(\omega)$ and $\mathbb{P}^*(\omega)$ are functions of $S_T(\omega)$, the terminal stock price.

The greedy algorithm gives very accurate results when applied in binomial model with realistic parameters and sufficient number of time steps. Indeed, in context of Problem 2 the error is by (13) and (26) bounded above by

$$p_{max} = (\max\{p, 1-p\})^T.$$

As regards Problem 4, the error is by (22) and (27) bounded above by cp^*_{max}, where

$$p^*_{max} = (\max\{p^*, 1-p^*\})^T.$$

In Table 6.1 we have listed the values for p_{max} for parameter values $p = 0.5, 0.6, 0.7, 0.8, 0.9$ and $T = 10, 20, 50, 100$. We see that p_{max} takes unsatisfactory values if p deviates a lot from 0.5 and the number of time steps is small. However, these cases can be considered as unrealistic. In particular, as we pointed out in [14], when binomial model is used to approximate the Black-Scholes model, the parameters p and p^* are close to 0.5.

Table 6.1: The quantity p_{max} for different values of p and T.

p	$p_{max}^{(10)}$	$p_{max}^{(20)}$	$p_{max}^{(50)}$	$p_{max}^{(100)}$
0.5	9.8×10^{-4}	9.5×10^{-7}	8.9×10^{-16}	7.9×10^{-31}
0.6	0.0060	3.7×10^{-5}	8.1×10^{-12}	6.5×10^{-23}
0.7	0.028	8.0×10^{-4}	1.8×10^{-8}	3.2×10^{-16}
0.8	0.11	0.012	1.4×10^{-5}	2.0×10^{-10}
0.9	0.35	0.12	0.0052	2.7×10^{-5}

6.2 The greedy algorithm in the binomial model

Consider now a European option with discounted payoff H and assume that the investor wants to find an approximately optimal solution for Problem 2 using the greedy algorithm. A general, but naive way to do this would be to first evaluate the value of the ratio

$$\frac{\mathbb{P}(\omega)}{\mathbb{Q}(\omega)} = \frac{\mathbb{P}(\omega)\mathbb{E}^*(H)}{H(\omega)\mathbb{P}^*(\omega)} \tag{28}$$

(cf. (11) and (12)) for each path $\omega \in \Omega$ separately, then order the paths so that the ratio $\mathbb{P}(\omega)/\mathbb{Q}(\omega)$ is decreasing, and finally, using this ranking, choose to hedge the option in as many scenarios as the cost constraint allows. However, the number of paths in a binomial model with T time steps is equal to 2^T, which makes the procedure described above computationally infeasible even for quite small values of T. Hence we see that although the greedy algorithm described in Section 3 looks simple, its implementation in binomial model is not completely trivial.

However, we have been able to develop efficient ways to implement the greedy algorithm for some special types of European options. In [12], pp. 442–447 we do this for simple options and in [13] for lookback options.

By a European simple option we mean an option whose payoff at time T depends only on the terminal stock price S_T. Thus, the discounted payoff of such an option can be written as

$$H = r^{-T} g(S_T), \tag{29}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic function. The most common example of such an option is probably the European call option with

$$H = r^{-T}(S_T - K)^+,$$

where K is the strike price of the option.

For options with payoff H in (29) the ratio in (28) is a function of S_T, since both $\mathbb{P}$ and $\mathbb{P}^*$ are functions of S_T in a binomial model. Thus, instead of ordering the 2^T individual paths we can order the $T + 1$ "S_T-atoms" $E_0, E_1, \ldots, E_T$, where

$$E_i := \{\omega \in \Omega : S_T(\omega) = S_0 u^i d^{T-i}\}$$

so that the ratio in (28) is decreasing, and using this ranking choose as many atoms as possible to the set of hedged scenarios. Finally, we can study the critical atom E_s, i.e. the first atom that we no longer can afford to choose wholly, separately to see how many paths belonging to E_s we can still afford to choose individually. For details we refer to [12], pp. 442–447.

Remark 11. *Recall that the greedy algorithm can be used also in connection with Problem 4 in Section 4.2 as well as the corresponding American problem described in Section 5.2. In this case the ratio according to which the paths should be ordered is $\mathbb{P}^*(\omega)/\mathbb{P}(\omega)$, which is a function of S_T. Therefore, the method for applying greedy algorithm for simple options developed in [12], pp. 442–447 can be used also in this case.*

In [13] we study the greedy algorithm in case of a European lookback option whose discounted payoff at time T is of the form

$$H = r^{-T} f(S_T, \max_{0 \leq i \leq T} S_i),$$

where $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic function. In this case the ratio in (28) is a function of the two quantities S_T and $\max_{0 \leq i \leq T} S_i$, which is why we can order the "$(S_T, \max_{0 \leq i \leq T} S_i)$-atoms" instead of the individual paths. Of the graphical scheme in [13] we see that the number of such atoms is $0.25T^2 + T + 0.75$, if T is odd and $0.25T^2 + T + 1$, if T is even. Thus, also in the case of lookback options it is possible to reduce the complexity of the greedy algorithm remarkably.

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